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MULTIVARIATE HERMITE POLYNOMIALS, GRAM-CHARLIER EXPANSIONS AND EDGEWORTH EXPANSIONS*

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1. Summary

In this paper multivariate Hermite polynomials are defined and applied to the problem of approximating certain multivariate cumulative distribution functions (c.d.f.'s) just as univariate Hermite polynomials are used in Gram-Charlier and Edgeworth expansions; the c.d.f.'s which are symmetric in their variables are of particular interest. The inverse problem of using these polynomials to approximate the percentage point corresponding to an arbitrary fixed value of the c.d.f. is also considered.

2. Introduction

Multivariate Hermite polynomials were already defined by Hermite [3] in 1864 and later studied by Appel and Kampé de Fériet [1]. The univariate case has been extensively used and applied in Gram-Charlier and Edgeworth expansions but the corresponding multivariate result appears to have had little application in the past. In this paper we develop the machinery needed to apply these multivariate Hermite polynomials to the approximation of certain probabilities. Of particular interest is the application to approximating the probability of a correct selection in a wide class of ranking and selection problems. A particular example (that uses this machinery) is the selection of the best Poisson population (or Poisson processes) treated by Sobel [4].

3. Definition of the Multivariate Hermite Polynomials

Let $x = (x_1,\ldots,x_p)'$ denote a p-dimensional column vector and $f(x;\rho)$

denote a given, arbitrary density in p variables; it is assumed that the associated chance variables $X_1,\dots,X_p$ are all standardized, i.e., $EX_i = 0$, $EX_i^2 = 1$, and that there is a common correlation ρ between X_i and X_j $(i \neq j;\ i,j=1,2,\dots,p)$ such that

$$\underline{(3.1)} \qquad -(p-1)^{-1} < \rho < 1$$

Let $\varphi(x;\rho)$ denote the corresponding standardized multivariate Normal density with a common correlation ρ between the Normal chance variables X_i^* and X_j^* $(i \neq j;\ i,j=1,2,\dots,p)$, where the value of ρ in $\varphi(x;\rho)$ is the same as in $f(x;\rho)$ so that (3.1) holds. If (3.1) holds, then the associated covariance (or correlation) matrix

$$\underline{(3.2)} \qquad \Sigma = A^{-1} = \{\sigma_{ij}\} = \{\rho+(1-\rho)\delta_{ij}\}$$

(δ_{ij} being the Kronecker delta) and its inverse $A = \Sigma^{-1}$ are positive definite.

Consider an expansion of $f(x;\rho)$ about $\varphi(x;\rho)$ in the form

$$\underline{(3.3)} \qquad f(x;\rho) = \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{\infty} (-1)^s \frac{C_{j_1,\dots,j_p}(\rho)}{j_1!\cdots j_p!} \varphi^{(j_1,\dots,j_p)}(x;\rho)$$

where $s=j_1+\dots+j_p$ and the superscript $(j_1,\dots,j_p)$ denotes a mixed partial derivative taken j_α times with respect to x_α $(\alpha=1,2,\dots,p)$. The multivariate Hermite polynomial is defined by

$$\underline{(3.4)} \qquad H^*_{j_1,\dots,j_p}(x;\rho) = \frac{(-1)^s}{\varphi(x;\rho)} \varphi^{(j_1,\dots,j_p)}(x;\rho)$$

so that we can write (3.3) as

$$\underline{(3.5)} \qquad f(x;\rho) = \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{\infty} \frac{C_{j_1,\dots,j_p}(\rho)}{j_1!\cdots j_p!} H^*_{j_1,\dots,j_p}(x;\rho)\, \varphi(x;\rho).$$

From (3.4) we find that the partial derivatives of the quadratic form $x'Ax$ with respect to x_j $(j=1,2,\ldots,p)$ play an essential role in simplifying the structure of these Hermite polynomials and this leads us to define new variables $y=Ax/b$ or equivalently

$$\underline{(3.6)} \qquad y_i = b(x_i + \rho' \sum_{\substack{j=1 \\ j \neq i}}^{p} x_j) \qquad (i=1,2,\ldots,p)$$

where

$$\underline{(3.7)} \qquad \rho' = \frac{-\rho}{1+(p-2)\rho} \; ; \qquad b = \sqrt{\frac{1+(p-2)\rho}{(1-\rho)[1+(p-1)\rho]}}$$

The scalar $b=b_p(\rho)$ has been defined so as to make the y_i standardized variables and they have a common correlation which is given by ρ' in (3.7); we note that ρ' also satisfies the inequalities in (3.1). We can now define the simpler polynomials $H_{j_1,\ldots,j_p}(x;\rho)$ by the relation

$$\underline{(3.8)} \qquad H^*_{j_1,\ldots,j_p}(x;\rho) = b_p^s \, H_{j_1,\ldots,j_p}(y;\rho') \qquad (j_1,\ldots,j_p = 0,1,\ldots).$$

For example it is easy to show (the proof is omitted) that if $H_{j_1,\ldots,j_p}(x;\rho)$ has all subscripts zero except possibly for j_r $(=j$, say) then

$$\underline{(3.9)} \qquad H_{0,\ldots,0,j,0,\ldots,0}(x;\rho) = H_j(y_r) \qquad (j=0,1,\ldots)$$

where $H_j(y)$ denotes the standard univariate Hermite polynomial in y of order j. Setting $p=1$ in (3.9), we note that $b=r=1$, $x=y_1$ and our notation agrees with the standard notation of Hermite polynomials. Because of this (and because the definition $y=Ax/b$ differs from the definition $\xi=Ax$ in [1]), our notation differs from that in [1]; it is easily seen from (3.10) that our H^* and H correspond to their H and G, respectively.

Letting $\varphi_1(y;\rho)$ denote the joint density of the y_i $(i=1,2,\ldots,p)$, we now define another set of polynomials $H^{**}_{j_1,\ldots,j_p}(y;\rho)$ for any non-negative integers $j_1,\ldots,j_p$ by the relation

$$\underline{(3.10)} \qquad H^{**}_{j_1,\ldots,j_p}(y;\rho) = \frac{(-1)^s}{\varphi_1(y;\rho)}\ \varphi_1^{(j_1,\ldots,j_p)}(y;\rho).$$

Since the y_i are standardized and equi-correlated with common correlation ρ' it follows that $\varphi_1(y;\rho) = \varphi(y;\rho')$ and hence a comparison of (3.4) and (3.10) gives

$$\underline{(3.11)} \qquad H^{**}_{j_1,\ldots,j_p}(y;\rho) = H^{*}_{j_1,\ldots,j_p}(y;\rho').$$

By (3.8), the right hand member of (3.11) can be written in terms of the simpler polynomials $H_{j_1,\ldots,j_p}(z;\rho'')$ where the z_i are new variables defined as in (3.6) by

$$\underline{(3.12)} \qquad z_i = b_p(\rho')\ \{y_i+\rho'' \sum_{\substack{j=1 \\ j\neq i}}^{p} y_j\} \qquad\qquad (i=1,2,\ldots,p).$$

and where

$$\underline{(3.13)} \qquad \rho'' = -\rho'/[1+(p-2)\rho'] = \rho\ ; \quad b(\rho') = b(\rho).$$

Writing $y = M_1x$ for (3.6) and $z = M_2y$ for (3.12) and using (3.7) and (3.13), we find that $M_2M_1 = I$, the identity matrix. Hence, from (3.8) and (3.11)

$$\underline{(3.14)} \qquad H^{**}_{j_1,\ldots,j_p}(y;\rho) = b^s\ \ H_{j_1,\ldots,j_p}(x;\rho) \qquad (j_1,\ldots,j_p=0,1,\ldots).$$

Hence the H*-polynomials and the H**-polynomials can both be expressed in terms of the simpler standardized H-polynomials with the appropriate argument.

4. Biorthogonality Property and its Applications

Hermite [2] (See Appel and Kampé de Fériet [1], p. 367) has shown the remarkable fact that the two sets of polynomials $H_{j_1,\ldots,j_p}(x;\rho)$ and $H^*_{j_1,\ldots,j_p}(x;\rho)$ are biorthogonal, which we now state as a theorem.

Theorem 1 (Hermite): For all non-negative integers $i_1,\ldots,i_p$ and $j_1,\ldots,j_p$

$$\text{(4.1)} \quad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} H^*_{i_1,\ldots,i_p}(x;\rho)\, H_{j_1,\ldots,j_p}(x;\rho)\, \varphi(x;\rho)dx_1\ldots dx_p = \begin{cases} \prod_{\alpha=1}^{p}(j_\alpha)! \\ 0 \end{cases}$$

where the upper result holds if $i_\alpha=j_\alpha$ for all $\alpha(\alpha=1,2,\ldots,p)$ and the lower result holds if they differ for at least one α.

By (3.6) the density $\varphi_1(y;\rho)$ of y has the form (letting primes denote transpose)

$$\text{(4.2)} \quad \varphi_1(y;\rho) = e^{-\frac{b^2}{2}(y'A^{-1}y)} \frac{b^p}{\sqrt{(2\pi)^p \det(A)}} .$$

Using (4.2) the definition (3.10) of the polynomials $H^{**}_{j_1,\ldots,j_p}(y;\rho)$ is easily seen to be equivalent to defining the latter by means of the multivariate Taylor expansion

$$\text{(4.3)} \quad e^{-\frac{b^2}{2}(y-h)'A^{-1}(y-h)} = e^{-\frac{b^2}{2}y'A^{-1}y} \sum_{j_p=0}^{\infty}\cdots\sum_{j_1=0}^{\infty} \frac{h_1^{j_1}\cdots h_p^{j_p}}{j_1!\cdots j_p!} H^{**}_{j_1,\ldots,j_p}(y;\rho)$$

where h is the vector $(h_1,h_2,\ldots,h_p)'$. Cancelling common factors in (4.3), letting t=bh, and using (3.14) and the fact that $x=bA^{-1}y$, we obtain the result

$$\text{(4.4)} \quad e^{-\frac{1}{2}t'A^{-1}t+t'x} = \sum_{j_p=0}^{\infty}\cdots\sum_{j_1=0}^{\infty} \frac{t_1^{j_1}\cdots t_p^{j_p}}{j_1!\cdots j_p!} H_{j_1,\ldots,j_p}(x;\rho)$$

If we now multiply the left members of (4.4) and (3.5) and also the right members, replace t by it, and integrate over the x's then, using the biorthogonality (4.1), we obtain

$$\underline{(4.5)} \qquad e^{\frac{1}{2}t'A^{-1}t}\,\psi(t) = \sum_{j_p=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{(it_1)^{j_1} \cdots (it_p)^{j_p}}{j_1! \cdots j_p!}\, C_{j_1,\ldots,j_p}(\rho),$$

where $\psi(t)$ is the characteristic function (c.f.) of the arbitrary, given multivariate density $f(x;\rho)$. This result is used in Section 6 to develop the Edgeworth expansion of $f(\xi;\rho)$ when the vector ξ is the sum of n independent and identically distributed vector random variables.

In writing out the standardized polynomials $H_{j_1,\ldots,j_p} = H_{j_1,\ldots,j_p}(x;\rho)$ for any fixed value of s we need only consider the cases $j_1 \geq j_2 \geq \ldots \geq j_p$ since all the others can be readily obtained by a permutation of the variables. For general p and s=0,1,2,3,4 and 6 these polynomials are given as follows:

$$\underline{(4.6)} \qquad H_{0,\ldots,0} = 1 \; ; \qquad H_{1,0,\ldots,0} = x_1$$

$$H_{2,0,\ldots,0} = x_1^2-1 \; ; \qquad H_{1,1,0,\ldots,0} = x_1x_2-\rho$$

$$H_{3,0,\ldots,0} = x_1^3-3x_1 \; ; \qquad H_{2,1,0,\ldots,0} = x_1^2x_2-x_2-2x_1\rho$$

$$H_{1,1,1,0,\ldots,0} = x_1x_2x_3-\rho(x_1+x_2+x_3)$$

$$H_{4,0,\ldots,0} = H_4(x_1) = x_1^4-6x_1^2+3$$

$$H_{3,1,0,\ldots,0} = (x_1^3-3x_1)x_2-3\rho(x_1^2-1)$$

$$H_{2,2,0,\ldots,0} = (x_1^2-1)(x_2^2-1)-4\rho(x_1x_2-\rho)-2\rho^2$$

$$H_{2,1,1,0,\ldots,0} = (x_1^2-1)(x_2x_3-\rho)-2\rho(x_1x_2-\rho)-2\rho(x_1x_3-\rho)-2\rho^2$$

$$H_{1,1,1,1,0,\ldots,0} = S_4(4)-\rho[S_4(2)-6\rho]-3\rho^2$$

$$H_{6,0,\ldots,0} = H_6(x_1) = x_1^6-15x_1^4+45x_1^2-15$$

$$H_{5,1,0,\ldots,0} = H_5(x_1)H_1(x_2)-5\rho H_4(x_1) \quad (\text{where } H_5(x_1) = x_1^5-10x_1^3+15x_1)$$

$$H_{4,2,0,\ldots,0} = H_4(x_1)H_2(x_2)-8\rho H_3(x_1)H_1(x_2)+12\rho^2H_2(x_1)$$

$$H_{3,3,0,\ldots,0} = H_3(x_1)H_3(x_2)-9\rho H_2(x_1)H_2(x_2)+18\rho^2H_1(x_1)H_1(x_2)-6\rho^3$$

$$H_{4,1,1,0,\ldots,0} = H_4(x_1)(x_2x_3-\rho)-4\rho H_3(x_1)[H_1(x_2)+H_1(x_3)]+12\rho^2H_2(x_1)$$

$$H_{3,2,1,0,\ldots,0} = H_3(x_1)H_2(x_2)H_1(x_3)-2\rho H_3(x_1)H_1(x_2)-3\rho H_2(x_1)H_2(x_2)$$
$$-6\rho H_2(x_1)(x_2x_3-\rho)+12\rho^2(x_1x_2-\rho)+12\rho^3$$

$$H_{2,2,2,0,\ldots,0} = H_2(x_1)H_2(x_2)H_2(x_3)-4\rho[H_2(x_1)(x_2x_3-\rho)+H_2(x_2)(x_1x_3-\rho)$$
$$+H_2(x_3)(x_1x_2-\rho)]+8\rho^2(x_1x_2+x_1x_3+x_2x_3-3\rho)-2\rho^2[H_2(x_1)$$
$$+H_2(x_2)+H_2(x_3)]+16\rho^3$$

$$H_{3,1,1,1,0,\ldots,0} = H_3(x_1)[x_2x_3x_4-\rho(x_2+x_3+x_4)]-3\rho H_2(x_1)(x_2x_3+x_2x_4$$
$$+x_3x_4-3\rho)+6\rho^2x_1(x_2+x_3+x_4)-6\rho^3$$

$$H_{2,2,1,1,0,\ldots,0} = H_2(x_1)H_2(x_2)(x_3x_4-\rho)-4\rho S_4(4)-2\rho H_2(x_1)[(x_2x_3-\rho)$$
$$+(x_2x_4-\rho)]-2\rho H_2(x_2)[(x_1x_3-\rho)+(x_1x_4-\rho)]-4\rho^2[S_4(2)-6\rho]$$
$$-2\rho^2[H_2(x_1)+H_2(x_2)]-2\rho^2(x_3x_4-\rho)+8\rho^2(x_1x_2-\rho)+20\rho^3$$

$$H_{2,1,1,1,1,0,\ldots,0} = H_2(x_1)[S_4'(4)-\rho(S_4'(2)-6\rho)-3\rho^2]-2\rho x_1S_4'(3)$$
$$+6\rho^2x_1S_4'(1)+2\rho^2[S_4'(2)-6\rho]$$

$$H_{1,1,1,1,1,1,0,\ldots,0} = S_6(6)-\rho S_6(4)+3\rho^2[S_6(2)-15\rho]+30\rho^3 .$$

In the above $S_4(j)$ denotes the sum of products of x's taken j at a time from the set (x_1,x_2,x_3,x_4); $S_4'(j)$ denotes the sum of products of x's taken j at a time from the set (x_2,x_3,x_4,x_5); and $S_6(j)$ denotes the sum of products of x's taken j at a time from the set $(x_1,x_2,x_3,x_4,x_5,x_6)$. The polynomials for $s \leq 4$ and $s = 6$ are given above because they appear in the constant term, the $n^{-\frac{1}{2}}$ term and the n^{-1} term in the Edgeworth expansion derived in Section 6; the polynomials for $s = 5$ are first used in the $n^{-3/2}$ term and, since these are not considered in Section 6, they are omitted here. Each expression above is meaningful and holds for any p equal to or greater than the number of non-zero subscripts on $H_{j_1,\ldots j_p}$; it should be noted that these expressions do not otherwise depend on p. To obtain the "starred" Hermite polynomials we use (3.6) and (3.8).

The derivations of (4.6) were obtained by using the following recursion formula (for convenience, we take $j_i = \alpha+1$ $(\alpha \geq 0)$ and write the recursion on the $i^{\underline{th}}$ subscript or variable without writing all the subscripts)

$$\underline{(4.7)} \quad H_{\ldots,\alpha+1,\ldots}(x;\rho) = \left\{ x_i - \frac{\partial}{\partial x_i} - \rho \sum_{\substack{j=1 \\ j \neq i}}^{p} \frac{\partial}{\partial x_j} \right\} H_{\ldots,\alpha,\ldots}(x;\rho)$$

To prove (4.7) we first note from (3.12) and (3.13) that $x = b\Sigma y$ so that

$$\underline{(4.8)} \qquad \frac{\partial \varphi(x;\rho)}{\partial y_i} = -\frac{b^2}{2}\, \varphi(y;\rho') \frac{\partial}{\partial y_i}(y'\Sigma y) = -b^2 \varphi(y;\rho')(y_i + \rho \sum_{\substack{j=1 \\ j \neq i}}^{p} y_j)$$

$$= -bx_i \varphi(y;\rho') = -bx_i \varphi(x;\rho).$$

Starting with left side of (4.7) and using (3.4) and the above, we have

$$\underline{(4.9)} \quad H_{\ldots,\alpha+1,\ldots}(x;\rho)\, \varphi(x;\rho) = b^{-s} H^*_{\ldots,\alpha+1,\ldots}(y;\rho')\, \varphi(y;\rho')$$

$$= (-b)^{-s}\, \varphi^{(\ldots,\alpha+1,\ldots)}(y;\rho') = (-b)^{-s} \frac{\partial}{\partial y_i} \varphi^{(\ldots,\alpha,\ldots)}(y;\rho')$$

$$= -(b^{-s}) \frac{\partial}{\partial y_i} \{ H^*_{\ldots,\alpha,\ldots}(y;\rho')\, \varphi(y;\rho') \}$$

$$= -b^{-1} \frac{\partial}{\partial y_i} \{ H_{\ldots,\alpha,\ldots}(x;\rho)\, \varphi(x;\rho) \} = -b^{-1} \varphi(x;\rho) [-bx_i H_{\ldots,\alpha,\ldots}(x;\rho)$$

$$+ \sum_{j=1}^{p} \left(\frac{\partial}{\partial x_j} H_{\ldots,\alpha,\ldots}(x;\rho) \right) \frac{\partial x_j}{\partial y_i}]$$

Since $x = b\Sigma y$ it follows that $\frac{\partial x_j}{\partial y_i} = b$ for $j=i$ and $b\rho$ for $j\neq i$. Substituting these in the last member of (4.9) yields the result (4.7).

If we multiply both members of (3.5) by $H_{j_1,\ldots,j_p}(x;\rho)$ and integrate out all the x_i $(i=1,2,\ldots,p)$ then $C_{j_1,\ldots,j_p}$ (a function of p, ρ and f) is given for all non-negative integers $j_1,\ldots,j_p$ by

$$\underline{(4.10)} \qquad C_{j_1,\ldots,j_p} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} H_{j_1,\ldots,j_p}(x;\rho)\ f(x)dx_1\ldots dx_p$$

It is clear that $C_{0,\ldots,0} = H_{0,\ldots,0} = H^*_{0,\ldots,0} = 1$. If, under $f(x;\rho)$, the X_i $(i=1,2,\ldots,p)$ all have zero mean then we see from (4.6) and (4.10) that $C_{1,0,\ldots,0} = C_{0,1,0,\ldots,0} = \cdots = C_{0,\ldots,0,1} = 0$. If, in addition, the X_i all have unit variance under $f(x;\rho)$ and there is a common correlation coefficient ρ between X_i and X_j $(i\neq j)$, which is the same as the value of ρ in $\varphi(x;\rho)$, then we find that all second order C-values, i.e., those with s=2, are also zero. In writing down the third and fourth order C-values it is only necessary to give them for $j_1 \geq j_2 \geq\ldots\geq j_p$ since the others are obtained by a permutation of the subscripts of the μ's in (4.11). Direct computation gives

$$C_{3,0,\ldots,0} = \mu_{3,0,\ldots,0}\ ; \quad C_{2,1,0,\ldots,0} = \mu_{2,1,0,\ldots,0}\ ;$$

$$C_{1,1,1,0,\ldots,0} = \mu_{1,1,1,0,\ldots,0}\ ;$$

$$\underline{(4.11)} \qquad C_{4,0,\ldots,0} = (\mu_{4,0} - 3)\ ; \quad C_{3,1,0,\ldots,0} = (\mu_{3,1,0,\ldots,0} - 3\rho)\ ;$$

$$C_{2,2,0,\ldots,0} = (\mu_{2,2,0,\ldots,0} - 2\rho^2-1)\ ; \quad C_{2,1,1,0,\ldots,0} = (\mu_{2,1,1,0,\ldots,0} - \rho-2\rho^2)\ ;$$

$$C_{1,1,1,1,0,\dots,0} = (\mu_{1,1,1,1,0,\dots,0} - 3\rho^2)$$

where $\mu_{j_1,\dots,j_p}$ is the expectation of $\prod_{\alpha=1}^{p} X_\alpha^{j_\alpha}$ under $f(x;\rho)$.

5. Integration of the Gram-Charlier Expansion

After computing the constants $C_{j_1,\dots,j_p}$ our next task is to obtain $F(x;\rho)$, which is defined as the c.d.f. of $f(x;\rho)$. In order to integrate (3.5) term by term from $-\infty$ to x_i in the "dummy" variable t_i $(i=1,2,\dots,p)$ we make use of the following auxiliary facts which are readily checked. From (3.4)

$$\text{(5.1)} \quad \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} H^*_{j_1,\dots,j_p}(t;\rho)\, \varphi(t;\rho) dt_1 \dots dt_p$$

$$= (-1)^p H^*_{j_1-1,\dots,j_p-1}(x;\rho)\, \varphi(x;\rho)$$

provided $j_\alpha \geqq 1$ for $\alpha=1,2,\dots,p$. More generally, if exactly r of the subscripts, say $i_1, i_2, \dots, i_r$ $(0 \leqq r < p)$, are positive then by (4)

$$\text{(5.2)} \quad \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} H^*_{j_1,\dots,j_p}(t;\rho)\, \varphi(t;\rho) dt_1 \dots dt_p$$

$$= (-1)^r \int_{-\infty}^{x_{r+1}} \cdots \int_{-\infty}^{x_p} H^*_{j_1-1,\dots,j_r-1,j_{r+1},\dots,j_p}(t';\rho)\, \varphi(t';\rho) dt_{r+1} \dots dt_p$$

where $t' = \{x_1,\dots,x_r,t_{r+1},\dots,t_p\}$. Then $\varphi(t';\rho)$ can be factored into $\varphi_r(t'';\rho)\, \varphi_{p-r}(t''';\rho_2)$ where the first factor is the marginal density of $t_1 t_2,\dots,t_r$ [evaluated at $t_i = x_i$ $(i=1,2,\dots,r)$] and the second factor is the conditional density of $t_{r+1},\dots,t_p$ given $t_1=x_1,\dots,t_r=x_r$. The variables $t_{r+1},\dots,t_p$ can be transformed into standardized variables without changing the form of the region of integration. The Hermite polynomial in the right hand member of (5.2) can be viewed as a Hermite polynomial in p-r variables and if these are written as a linear combination of appropriate new variables, then the same method as used in (5.1) and (5.2) can be used again to lower by at least one the number of variables not yet integrated out; the only exceptions to this are those integrals in which the Hermite polynomial reduces to a constant before the integrations are all completed. These terms are handled by using the result (in terms of the scalar w).

$$\underline{(5.3)}\quad \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_s} \varphi(t;\rho)dt_1,\ldots,dt_s = \int_{-\infty}^{\infty}\left[\prod_{j=1}^{s}\Phi\left(\frac{w\sqrt{\rho}+x_j}{\sqrt{1-\rho}}\right)\right]\varphi(w)dw$$

where $\Phi(y)$ denotes the standardized univariate Normal c.d.f.; for $x_1=x_2=\ldots=x_s$ (=h, say) tables of this integral are available as a function of s, h and ρ. The proof of (5.3) lies in the fact that the left member of (5.3) can be written in terms of standardized independent Normal random variables W_j $(j=0,1,\ldots,s)$ as

$$\underline{(5.4)}\quad P\{\sqrt{1-\rho}\,W_j-\sqrt{\rho}\,W_o < x_j\ (j=1,\ldots,s)\}$$

$$= \int_{-\infty}^{\infty} P\{W_j < \frac{w\sqrt{\rho}+x_j}{\sqrt{1-\rho}}\ (j=1,\ldots,s)\}\ \varphi(w)dw$$

and the right hand member of (5.3) then follows.

For p=2 we give below the third and fourth order terms (or 1st and 2nd order correction terms) of order $1/\sqrt{n}$ and $1/n$, respectively; for $p \geqq 3$ only the third order terms of order $1/\sqrt{n}$ are given. For p=2, using (3.4), (3.5), (3.6) and (5.3) we obtain

$$\underline{(5.5)}\quad F(x;\rho) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f(t;\rho)dt_1dt_2 = \int_{-\infty}^{\infty}\left[\prod_{j=1}^{2}\Phi\left(\frac{w\sqrt{\rho}+x_j}{\sqrt{1-\rho}}\right)\right]\varphi(w)dw$$

$$+ \{[C_{3,0}Q(x_1,x_2) + C_{0,3}Q(x_2,x_1)] + [C_{2,1}R(x_1,x_2) + C_{1,2}R(x_2,x_1)]\}$$

$$+ \{[C_{4,0}S(x_1,x_2) + C_{0,4}S(x_2,x_1)] + [C_{3,1}T(x_1,x_2) + C_{1,3}T(x_2,x_1)]$$

$$+ C_{2,2}U(x_1,x_2)\} + \ldots$$

where

$$\underline{(5.6)} \quad Q(x_1,x_2) = -\frac{\varphi(x_1)}{1-\rho^2}\{(x_1^2-1)(1-\rho^2)\,\Phi(y_2) + [2\rho x_1\sqrt{1-\rho^2} - \rho^2 y_2]\,\varphi(y_2)\}$$

$$\underline{(5.7)} \quad R(x_1,x_2) = \frac{y_1\,\varphi(x_1)\,\varphi(y_2)}{1-\rho^2}$$

$$\underline{(5.8)} \quad S(x_1,x_2) = -\frac{\varphi(x_1)}{(1-\rho^2)^{3/2}}\{(1-\rho^2)^{3/2}\,(x_1^3-3x_1)\,\Phi(y_2)$$

$$+ 3\rho(1-\rho^2)(x_1^2-1)\,\varphi(y_2) - 3\rho^2\sqrt{1-\rho^2}\,x_1 y_2\varphi(y_2) + \rho^3(y_2^2-1)\,\varphi(y_2)\}$$

$$\underline{(5.9)} \quad T(x_1,x_2) = \frac{(y_1^2-1)}{(1-\rho^2)^{3/2}}\,\varphi(x_1)\,\varphi(y_2)$$

$$\underline{(5.10)} \quad U(x_1,x_2) = \frac{(y_1y_2+\rho)}{(1-\rho^2)^{3/2}}\,\varphi(x_1)\,\varphi(y_2).$$

In the above, y_1 and y_2 are regarded as functions of x_1 and x_2 through (3.6) in perfoming the indicated permutations. In particular applications of interest we may have $x_1=x_2$ so that $y_1=y_2$ and the given function $f(x_1,x_2)$ may be symmetric in its two arguments so that $C_{i,j}=C_{j,i}$ $(i,j=1,2,\ldots)$ and further simplification is then possible.

For p=3, we use the auxiliary variables y_i $(i=1,2,3)$, whose definition in (3.6) depends on p, and also

$$\underline{(5.11)} \qquad u_{ij} = \frac{x_i-\rho x_j}{\sqrt{1-\rho^2}} \quad ; \qquad v_{ij;m} = \frac{u_{im}-\rho(1-\rho)u_{jm}}{\sqrt{(1-\rho+\rho^2)(1+\rho-\rho^2)}}$$

for i, j and m all different. Then, by a similar procedure to that used for $p=2$, we obtain

$$\underline{(5.12)}\quad F(x;\rho) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3} f(t;\rho)dt_1dt_2dt_3 = \int_{-\infty}^{\infty}\left[\prod_{j=1}^{3}\Phi\left(\frac{w\sqrt{\rho}+x_j}{\sqrt{1-\rho}}\right)\right]\varphi(w)dw$$

$$+ \{[C_{3,0,0}Q(x_1,x_2,x_3) + C_{0,3,0}Q(x_2,x_1,x_3) + C_{0,0,3}Q(x_3,x_2x_1)]$$

$$+ [C_{2,1,0}R(x_1,x_2,x_3) + C_{2,0,1}R(x_1,x_3,x_2) + C_{1,2,0}R(x_2,x_1,x_3)$$

$$+ C_{0,2,1}R(x_3,x_1,x_2) + C_{0,1,2}R(x_3,x_2,x_1) + C_{1,0,2}R(x_2,x_3,x_1)]$$

$$+ C_{1,1,1}S(x_1,x_2,x_3)\} + \ldots$$

where

$$\underline{(5.13)}\quad Q(x_1,x_2,x_3) = \varphi(x_1)\ \{a^2(\rho)[u_{2,1}\varphi(u_{2,1})\ \Phi(v_{3,2,1})$$

$$+ u_{3,1}\varphi(u_{3,1})\ \Phi(v_{2,3,1})] + 2a^2(\rho)\sqrt{\frac{1+\rho-\rho^2}{1-\rho+\rho^2}}\,\varphi(u_{2,1})\ \varphi(v_{3,2,1})$$

$$+ \frac{2x_1\ a(\rho)}{\sqrt{(1-\rho+\rho^2)(1+\rho-\rho^2)}}[\varphi(u_{2,1})\ \Phi(v_{3,2,1}) + \varphi(u_{3,1})\ \Phi(v_{2,3,1})]$$

$$+ [x_1^2-d(\rho)]\int_{-\infty}^{\infty}\left[\prod_{j=2}^{3}\Phi\left(\frac{w\sqrt{\rho(1-\rho)}+u_{j1}}{\sqrt{1-\rho+\rho^2}}\right)\right]\varphi(w)dw\}.$$

$$\underline{(5.14)}\quad a(\rho) = \frac{\rho(1+\rho-\rho^2)}{1+2\rho}\sqrt{\frac{1+\rho}{1-\rho}}\ ;\qquad d(\rho) = \frac{(1+\rho)(1+2\rho-2\rho^2-2\rho^3+2\rho^4)}{(1-\rho)(1+2\rho)^2}\ .$$

$$\underline{(5.15)}\quad R(x_1,x_2,x_3) = \frac{\varphi(x_1)\ \varphi(u_{2,1})}{1-\rho^2}\left[u_{2,1}\ \Phi(y_3) + \frac{\rho\varphi(y_3)}{\sqrt{1+2\rho}}\right]$$

$$\underline{(5.16)}\quad S(x_1,x_2,x_3) = -\frac{\varphi(x_1)\ \varphi(u_{2,1})\ \varphi(y_3)}{(1-\rho)\sqrt{1+2\rho}}\ .$$

In the above the y_i, $u_{i,j}$ and $v_{i,j;m}$ are all regarded as functions of the x_i in performing the permutations indicated in (5.12); these permutations can be carried out on the subscripts of y_i, $u_{i,j}$ and $v_{i,j;m}$.

Let $t_{i;j,m}$ denote the value of y_i in (3.6) with p fixed at 3 and the other two x's equal to x_j and x_m. Consider the problem of getting the first correction term for general $p \geq 3$. The above formulas still hold, the only changes being that i) in place of y_3 in (5.15) and (5.16) we would write $t_{3;1,2}$ (the last two subscripts correspond to x_1 and x_2 on the left side of (5.15) and (5.16) and ii) the number of Q-terms, R-terms and S-terms in (5.12) is given by the multinomial coefficients $\binom{p}{1}$, $\frac{p!}{1!1!(p-2)!}$ and $\binom{p}{3}$, respectively. Permutations can now be carried out on the subscripts of $u_{i,j}$, $v_{i,j;m}$ and $t_{i;j,m}$.

6. Edgeworth Expansion

In this section we will assume that the matrix $\Sigma=A^{-1}$ is any positive definite covariance matrix and not necessarily a special correlation matrix as in Section 3. It is also assumed that the characteristic function $\psi(t)$ associated with the given density $f(x)$ is absolutely integrable so that (10.6.3) of [2] holds. The discussion and the notation used below follow those used by Cramér [2] for the univariate case. If Σ is any positive definite matrix the multivariate Hermite polynomials are defined as in (3.4) but we will then write $\varphi(x;\Sigma)$ and $H^*(x;\Sigma)$ instead of $\varphi(x;\rho)$ and $H^*(x;\rho)$, respectively. Explicit expressions for $H^*(x;\Sigma)$ for p=2 and $s \leq 3$ are given in [1].

Let $\xi_j=(\xi_{j1},\xi_{j2},\ldots,\xi_{jp})$ for each j (j=1,2,...,n) denote a p-dimensional vector of random variables which has a mean vector $\mu=(\mu_1,\mu_2,\ldots,\mu_p)$ and a positive definite covariance matrix Σ. The vectors $\xi_1,\xi_2,\ldots,\xi_n$ are assumed to be independently and identically distributed and we let ξ denote their sum. The characteristic function (c.f.) $\psi(t)$ of $(\xi-n\mu)/\sqrt{n} = \xi^*$ (say) in terms of the c.f. $\psi_1(t)$ of $\xi_1-\mu$ is given by

$$\underline{(6.1)} \quad \psi(t) = \left[\psi_1\left(\frac{t}{\sqrt{n}}\right)\right]^n = \exp\left\{n \sum_{j_p=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{\lambda'_{j_1,\ldots,j_p}}{j_1!\cdots j_p!}\left(\frac{it_1}{\sqrt{n}}\right)^{j_1}\cdots\left(\frac{it_p}{\sqrt{n}}\right)^{j_p}\right\}$$

where $\lambda'_{j_1,\ldots,j_p}$ are by definition the semi-invariants (or cumulants) of $\xi_1-\mu$. The first term with $s=j_1+\ldots+j_p=0$ is zero and, since all the $\xi_j-\mu$ have mean zero, the p terms with s=1 are also zero. It then follows from (6.1) that for s=2 the cumulants are identical with the variances and covariances of $\xi_1-\mu$ and hence these terms must add up to yield $-\frac{1}{2}t'\Sigma t$. Hence we can write (6.1) as

$$\underline{(6.2)} \quad e^{\frac{1}{2}t'\Sigma t}\,\psi(t)=\exp\left\{\sum_{s=3}^{\infty} \sum_{\substack{j_p=0 \\ j_1+\ldots+j_p=s}}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{\lambda'_{j_1,\ldots,j_p}}{j_1!\cdots j_p!}\,\frac{(it_1)^{j_1}\cdots(it_p)^{j_p}}{n^{(s/2)-1}}\right\}$$

Since the covariance matrix Σ of ξ^* is the same as for $\xi_1-\mu$ we can now find the relations between the C-values for ξ^* and the λ'-values for $\xi_1-\mu$ by expanding the right hand member of (6.2) and comparing term by term with the expansion in (4.5); we note in particular that for $3 \leq s \leq 5$

$$\underline{(6.3)} \quad C_{j_1,\ldots,j_p} = \frac{\lambda'_{j_1,\ldots,j_p}}{n^{(s/2)-1}}$$

and that for $s \geq 6$ there are at least two terms on the right.

If we now expand the right hand side of (6.2) in powers of $n^{-\frac{1}{2}}$, we obtain

$$\underline{(6.4)} \quad \psi(t) = e^{-\frac{1}{2}t'\Sigma t}\left\{\sum_{\alpha=0}^{\infty} \frac{1}{\alpha!}\left[\sum_{s=3}^{\infty} \sum_{j_p=0}^{\infty} \cdots \sum_{j_1=0}^{\infty} \frac{\lambda'_{j_1,\ldots,j_p}}{j_1!\cdots j_p!}\,\frac{(it_1)^{j_1}\cdots(it_p)^{j_p}}{n^{(s/2)-1}}\right]^{\alpha}\right\}$$

$$= e^{-\frac{1}{2}t'\Sigma t}\Bigg\{1+\frac{1}{\sqrt{n}}\Bigg[\Sigma\frac{\lambda'_{3,0,\ldots,0}}{6}(it_1)^3+\Sigma\frac{\lambda'_{2,1,0,\ldots,0}}{2}(it_1)^2(it_2)$$

$$+\Sigma\lambda'_{1,1,1,0,\ldots,0}(it_1)(it_2)(it_3)\Bigg]+\frac{1}{n}\Bigg[\Sigma\frac{\lambda'_{4,0,\ldots,0}}{24}(it_1)^4$$

$$+\Sigma\frac{\lambda'_{3,1,0,\ldots,0}}{6}(it_1)^3(it_2)+\Sigma\frac{\lambda'_{2,2,0,\ldots,0}}{4}(it_1)^2(it_2)^2$$

$$+\Sigma\frac{\lambda'_{2,1,1,0,\ldots,0}}{2}(it_1)^2(it_2)(it_3)$$

$$+\Sigma\lambda'_{1,1,1,1,0,\ldots,0}(it_1)(it_2)(it_3)(it_4)+\frac{1}{2}\Bigg(\Sigma\frac{\lambda'_{3,0,\ldots,0}}{6}(it_1)^3$$

$$+\Sigma\frac{\lambda'_{2,1,0,\ldots,0}}{2}(it_1)^2(it_2)+\Sigma\lambda'_{1,1,1,0,\ldots,0}(it_1)(it_2)(it_3)\Bigg)^2\Bigg]$$

$$+\mathcal{O}\left(\frac{1}{n^{3/2}}\right)\Bigg\}$$

where each unmarked summation indicates an appropriate sum over permutations and different positions of the non-zero subscripts of the λ' with a corresponding change in the subscripts of the t_i.

It is well known (see, for example, [2]) that the multivariate Normal distribution is sufficiently regular so that

i) the density can be regained directly from the c.f. as in (10.6.3) of [2] without first getting the c.d.f.

ii) the c.f. can be differentiated j_α times with respect to t_α for any positive integers j_α (α=1,2,...,p).

Hence it follows from the definition of the multivariate Hermite polynomial for any positive definite matrix Σ (as in (3.4)) that

$$\underline{(6.5)} \quad H^*_{j_1,\ldots,j_p}(x;\Sigma)\,\varphi(x;\Sigma) = (-1)^s\,\varphi^{(j_1,\ldots,j_p)}(x;\Sigma)$$

$$= \frac{(-1)^s\,\partial^s}{\partial x_1^{j_1}\cdots\partial x_p^{j_p}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}t'\Sigma t - it'x}}{(2\pi)^p}\,dt_1,\ldots,dt_p$$

$$= \frac{1}{(2\pi)^p}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}t'\Sigma t - it'x}(it_1)^{j_1}\cdots(it_p)^{j_p}\,dt_1\cdots dt_p.$$

Multiplying both extreme members of (6.4) by $(2\pi)^{-1}\exp\{-it'x\}$ and integrating out all the t_i gives the desired result

$$\begin{aligned}
\underline{(6.6)} \quad f(x) = \varphi(x;\Sigma) &+ \frac{\varphi(x;\Sigma)}{\sqrt{n}}\Big\{\Sigma\,\frac{\lambda'_{3,0,\ldots,0}}{6}H^*_{3,0,\ldots,0}(x;\Sigma) \\
&+ \Sigma\,\frac{\lambda'_{2,1,0,\ldots,0}}{2}H^*_{2,1,0,\ldots,0}(x;\Sigma) \\
&+ \Sigma\,\lambda'_{1,1,1,0,\ldots,0}H^*_{1,1,1,0,\ldots,0}(x;\Sigma)\Big\} \\
&+ \frac{\varphi(x;\Sigma)}{n}\Big\{\Sigma\,\frac{\lambda'_{4,0,\ldots,0}}{24}H^*_{4,0,\ldots,0}(x;\Sigma) \\
&+ \Sigma\,\frac{\lambda'_{3,1,0,\ldots,0}}{6}H^*_{3,1,0,\ldots,0}(x;\Sigma) \\
&+ \Sigma\,\frac{\lambda'_{2,2,0,\ldots,0}}{4}H^*_{2,2,0,\ldots,0}(x;\Sigma) \\
&+ \Sigma\,\frac{\lambda'_{2,1,1,0,\ldots,0}}{2}H^*_{2,1,1,0,\ldots,0}(x;\Sigma) \\
&+ \Sigma\,\lambda'_{1,1,1,1,0,\ldots,0}H^*_{1,1,1,1,0,\ldots,0}(x;\Sigma)
\end{aligned}$$

$$+ \frac{1}{2}\Bigg(\Sigma \frac{\lambda'_{3,0,\ldots,0}}{6} H^*_{3,0,\ldots,0}(x;\Sigma)$$

$$+ \Sigma\lambda'_{2,1,0,\ldots,0} H^*_{2,1,0,\ldots,0}(x;\Sigma)$$

$$+ \Sigma\,\lambda'_{1,1,1,0,\ldots,0} H^*_{1,1,1,0,\ldots,0}(x;\Sigma)\Bigg)^2 + \mathcal{O}\left(\frac{\varphi(x;\Sigma)}{n^{3/2}}\right) .$$

The operation of squaring in the last part of (6.6) is "symbolic" with respect to the H*-polynomials (not with respect to the coefficients λ') in the sense that the product of any two H*-polynomials as well as the square of any one is to be replaced by a single H*-polynomial in accordance with the rule

$$\underline{(6.7)} \quad H^*_{i_1,\ldots,i_p}(x;\Sigma)\, H^*_{j_1,\ldots,j_p}(x;\Sigma) \rightarrow H^*_{i_1+j_1,\ldots,i_p+j_p}(x;\Sigma) .$$

Finally the cumulants λ' can be replaced by the C-values of ξ^* using (6.3) and these in turn can be replaced by the central moments of ξ^* using (4.8); this may be more useful but it removes the powers of $n^{-\frac{1}{2}}$ from evidence. Alternatively we can replace the λ' by the C-values of $\xi_1-\mu$ and the latter by central moments of $\xi_1-\mu$ using (4.8), thus keeping the powers of $n^{-\frac{1}{2}}$ in evidence. The various powers of σ_i^2, the variance of the $i^{\underline{th}}$ component, $(i=1,2,\ldots,p)$ corresponding to the powers of σ^2 in (17.7.3) of [2], enter through the Hermite polynomials. The final result in terms of the Hermite polynomials is straightforward and is similar to that in (6.8) below except that $\varphi(x;\Sigma)$ and $H^*(x;\Sigma)$ are written instead of $\varphi(x;\rho)$ and $H^*(x;\rho)$, respectively.

In the special case when $\xi_1-\mu$ has variance unity for each of its components and a common correlation ρ (with $-(p-1)^{-1} < \rho < 1$) between the $\alpha^{\underline{th}}$ and $\beta^{\underline{th}}$ component $(\alpha \neq \beta)$ then Σ has the same structure as in Section 2 and the expressions in (4.6), with the help of (3.6), (3.7) and (3.8), can be used to give explicit results for the Edgeworth expansion. The final result in terms of the Hermite polynomials is

$$\underline{(6.8)} \quad f(x;\rho) = \varphi(x;\rho) + \frac{\varphi(x;\rho)}{\sqrt{n}} \Bigg\{ \Sigma \frac{\mu_{3,0,\ldots,0}}{6} H^*_{3,0,\ldots,0}(x;\rho)$$

$$+ \Sigma \frac{\mu_{2,1,0,\ldots,0}}{2} H^*_{2,1,0,\ldots,0}(x;\rho)$$

$$+ \Sigma \mu_{1,1,1,0,\ldots,0} H^*_{1,1,1,0,\ldots,0} \Bigg\}$$

$$+ \frac{\varphi(x;\rho)}{n} \Bigg\{ \Sigma \frac{(\mu_{4,0,\ldots,0} - 3)}{24} H^*_{4,0,\ldots,0}(x;\rho)$$

$$+ \Sigma \frac{(\mu_{3,1,0,\ldots,0} - 3\rho)}{6} H^*_{3,1,0,\ldots,0}(x;\rho)$$

$$+ \Sigma \frac{(\mu_{2,2,0,\ldots,0} - 2\rho^2 - 1)}{4} H^*_{2,2,0,\ldots,0}(x;\rho)$$

$$+ \Sigma \frac{(\mu_{2,1,1,0,\ldots,0} - \rho - 2\rho^2)}{2} H^*_{2,1,1,0,\ldots,0}(x;\rho)$$

$$+ \Sigma (\mu_{1,1,1,1,0,\ldots,0} - 3\rho^2) H^*_{1,1,1,1,0,\ldots,0}(x;\rho)$$

$$+ \tfrac{1}{2} \Bigg[\Sigma \frac{\mu_{3,0,\ldots,0}}{6} H^*_{3,0,\ldots,0}(x;\rho)$$

$$+ \Sigma \frac{\mu_{2,1,0,\ldots,0}}{2} H^*_{2,1,0,\ldots,0}(x;\rho)$$

$$+ \Sigma \mu_{1,1,1,0,\ldots,0} H^*_{1,1,1,0,\ldots,0}(x;\rho) \Bigg]^2 \Bigg\} + \mathcal{O}\left(\frac{\varphi(x;\rho)}{n^{3/2}}\right)$$

where the $\mu_{j_1,\ldots,j_p}$ represent mixed central moments of the components of $\xi_1-\mu$, i.e.,

$$\underline{(6.9)} \quad \mu_{j_1,\ldots,j_p} = E\left\{(\xi_{11}-\mu_1)^{j_1} \ldots (\xi_{1p}-\mu_p)^{j_p}\right\},$$

and the products and powers of the H*-polynomials arising from the squaring of the bracketed terms in (6.8) are "symbolic" and are to be replaced in accordance with the rule (6.7).

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REFERENCES

[1] Appel, P. and Kampé de Fériet, J. (1926). Fonctions Hypergéométriques et Hypersphériques, Gauther. Villars. Paris.

[2] Cramer, H. (1946). Mathematical Methods of Statistics, Princeton University Press. 133.

[3] Hermite, Ch. (1864) Comptes Rendus, 68 93-266 or Oeuvres II 293-308.

[4] Sobel, M. Single Sample Ranking Problems with Poisson Populations. Submitted to Ann. Math. Stat.